DISPERSION OF A MATERIAL IN AN OLDROYD LIQUID

K. V. Mukuk

UDC 532.593:532.529.5

A solution is obtained for the distribution of a trace component in a Oldroyd liquid for various forms of the pressure function: pulsed, stepwise, and periodic.

The model of [1, 2] for the dispersion of a material in a flow of viscous liquid is applicable also to laminar and turbulent flows provided that the diffusion processes are fairly prolonged. The use of this model is restricted to the range $W_m R/D \geq 6.9$ for the time factor, but this is not always obeyed for real processes (particularly ones of nonstationary type). In [3, 4], this restriction does not apply; they deal with nonstationary diffusion for any instant. For a rheologically complex medium [5] for which the relaxation time is $\lambda = 10^{-3}$–10^{0} sec, the duration of the diffusion processes are comparable with the transient-response times. Under those conditions, the diffusion processes will be dependent not only on the rheological ones [6] but also on the relaxation characteristics.

We consider the distribution of an impurity in a flow of Oldroyd liquid [7] with relaxation behavior:

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial\tau}{\partial t}=\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial\dot{\gamma}}{\partial t}. \tag{1}$$

The one-dimensional equation of motion in a circular tube then takes the form

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t}=\left(1+\lambda_1\frac{\partial}{\partial t}\right)f(t)+\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right). \tag{2}$$

The function $f(t) = (-1/\rho)(\partial p/\partial x)$ is dependent on time. We assume that $f(t)$ is one of the commoner functions: $f_1(t) = A\delta(t)$ is a Dirac delta function ($k = 1$), $f_2(t) = AH(t)$ a Heaviside step function ($k = 2$), and $f_3(t) = A\cos\omega t$ a harmonic function ($k = 3$). Then the velocity distribution at any instant is defined as

$$w(r,\ t)=\sum_{i=1}^{\infty}a_i^k(t)\,J_0(r,\ \xi)\ (k=1,\ 2,\ 3). \tag{3}$$

Here

$$a_i^1=\frac{2}{R\xi_iJ_1(R,\ \xi_i)\,\lambda_1(\alpha-\beta)}\left[(1+\lambda_1\alpha)\exp(\alpha t)-(1+\lambda_1\beta)\exp(\beta t)\right];$$

$$a_i^2=\frac{2A}{R\xi_iJ_1(R\xi_i)\,\lambda_1(\alpha-\beta)}\left[\frac{\exp(\alpha t)}{\alpha}-\frac{\exp(\beta t)}{\beta}+\frac{\alpha-\beta}{\alpha\beta}+\right.$$
$$\left.+\lambda_1(\exp(\alpha t)-\exp(\beta t))\right];$$

$$a_i^3=\frac{2A}{R\xi_iJ_1(R\xi_i)\,\lambda_1(\alpha-\beta)}\left[\frac{(1+\lambda_1\alpha)(\alpha\exp(\alpha t)-\alpha\cos\omega t)}{\alpha^2+\omega^2}+\right.$$
$$\left.+\frac{\omega\sin\omega t}{\alpha^2+\omega^2}-\frac{(1+\lambda_1\beta)(\beta\exp(\beta t)-\beta\cos\omega t+\omega t+\omega\sin\omega t)}{\beta^2+\omega^2}\right]\Big/2\lambda_1;$$

$$\beta=\left[-(1+\nu\lambda_2\xi_i^2)-\sqrt{(1+\nu\lambda_2\xi_i^2)^2-4\lambda_1\nu\xi_i^2}\right]/2\lambda_1;$$

$$\alpha=\left[-(1+\nu\lambda_2\xi_i^2)+\sqrt{(1+\nu\lambda_2\xi_i^2)^2-4\lambda_1\nu\xi_i^2}\right]/2\lambda_1;$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 42, No. 3, pp. 408-412, March, 1982. Original article submitted December 2, 1980.

0022-0841/82/4203-0292$07.50

and ξ_i are the roots of the equation for the Bessel function $J_0(R\xi_i) = 0$. The mean instantaneous flow speed is

$$W_m(t) = \frac{2}{R^2}\int_0^R w(r,\ t)\,rdr = \frac{2}{R}\sum_{i=1}^{n}\frac{a_i^k(t)}{\xi_i}J_1(R\xi_i). \tag{4}$$

If a trace of material is injected into the flow at the initial instant, one can determine the concentration c(t, x, r) at any instant by using the usual equation of convective diffusion [8]:

$$\frac{\partial c}{\partial t} + w(r,\ t)\frac{\partial c}{\partial x} = \frac{D_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right). \tag{5}$$

The following is the balance equation for matter in a coordinate system moving at the mean flow speed $W_m(t)$:

$$\pi R^2\frac{\partial c_m}{\partial t} + \frac{\partial Q}{\partial x} = 0,\ c_m = c_m(t,\ x) = \frac{2}{R^2}\int_0^R rcdr, \tag{6}$$

$$Q = 2\pi\int_0^R [w(r,\ t) - W_m(t)]\,rcdr. \tag{7}$$

We can take Taylor's assumption as correct for the instantaneous values of the time and write (5) as

$$[w(r,\ t) - W_m(t)]\frac{\partial c_m}{\partial x} = \frac{D_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right). \tag{8}$$

The left side of (8) is dependent on the variable r, and it is therefore simple to integrate it in order to determine the concentration profile in terms of the mean concentration over the flow section $\partial c_m/\partial t$; we get

$$c(t,\ x,\ r) = c_0(t,\ x) + \frac{\partial c_m}{\partial x}\int_0^r \frac{dr'}{D_0 r'}\int_0^{r'}[w(r,\ t) - W_m(t)]\,rdr. \tag{9}$$

From (9) we transform (3) to

$$Q = -\pi R^2 D\frac{\partial c_m}{\partial x}, \tag{10}$$

where the effective diffusion coefficient D is determined as

$$D = \frac{2}{D_0 R^2}\int_0^R \frac{dr'}{r}\left[\int_{r'}^R [w(r,\ t) - W_m(t)]rdr\right]^2. \tag{11}$$

Use of the velocity patterns of (3) and (4) for flows in relaxing systems in (11) for various pressure functions enables one to find D; here one has to incorporate the condition for orthogonality of the eigenfunctions $J_0(\xi_i,\ r)$ and use the major properties of Bessel functions [9]:

$$\int_0^R r^2 J_1(r\xi_i)\,dr = \frac{2R}{\xi_i^2}J_1(R\xi_i),\quad \int_0^R rJ_1(r\xi_i)\,J_1(r\xi_j)\,dr = \frac{R^2}{2}[J_1(R\xi_i)]^2.$$

It is characteristic that the effective diffusion coefficient is dependent not only on time but also on the relaxation characteristics: the relaxation time and the retardation time:

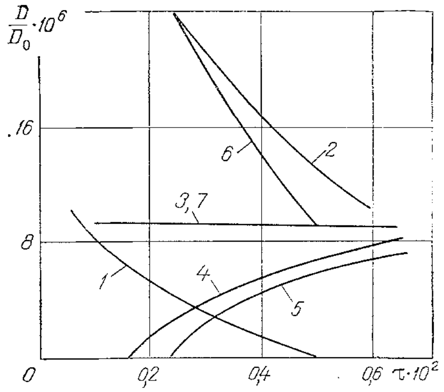

Fig. 1. Variation in the relative effective diffusion coefficient with time for various pressure functions: pulse (1); stepped ($\lambda_1 \neq \lambda_2 \neq 0$) (2), and ($\lambda_1 = \lambda_2 = 0$) (3), sinusoidal ($\omega = 1\ \mathrm{sec}^{-1}$, $\lambda_1 \neq \lambda_2 \neq 0$) (4), ($\omega = 1\ \mathrm{sec}^{-1}$, $\lambda_1 = \lambda_2 = 0$) (5), ($\omega = 0.01\ \mathrm{sec}^{-1}$, $\lambda_1 \neq \lambda_2 = 0$) (6), ($\omega = 0.01\ \mathrm{sec}^{-1}$, $\lambda_1 = \lambda_2 = 0$) (7).

$$D = \frac{2}{D_0R^2}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i^k(t)\,a_j^k(t)\,J_1(\xi_iR)\,J_1(\xi_jR)\left[\frac{R^2}{4\xi_i\xi_j} - \frac{2}{\xi_i\xi_j^3} - \frac{2}{\xi_i^3\,\xi_j}\right] + \sum_{i=1}^{\infty}[a_i^k(t)]^2\frac{R^2}{2\xi_i^2}\,J_1(\xi_iR)\right\} \quad (12)$$

or

$$D = \frac{2}{D_0R^2}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i^k(t)\,a_j^k(t)\left[\frac{R^2}{4\xi_i^2\,\xi_j^2} - \frac{2}{\xi_i\xi_j^3} - \frac{2}{\xi_i^3\,\xi_j}\right] + \sum_{i=1}^{\infty}[a_i^k(t)]^2\frac{R^2}{2\xi_i^2}\right. . \quad (13)$$

In the limit where the relaxation behavior of the liquid is so slight by comparison with the duration of the transient state, we have $\lambda_1 \to 0$, $\lambda_2 \to 0$, $\lambda_1(\alpha - \beta) \to 1$, $\alpha \to -\nu\mu^2/R^2$, $\beta \to -\infty$, $\mu_i = \xi_iR$, and the coefficients $a_i^k(t)$ in (13) take the form

$$a_i^1 = 2A\exp\left[-\nu\mu_iR^{-2}t/\mu_i\right],$$

$$a_i^2 = \frac{2A}{\mu_i}\left[\frac{\exp(-\nu\mu_it/R^2)}{-\nu\mu_i^2/R^2} - \frac{R^2}{\nu\mu_i^2}\right],$$

$$a_i^3 = \frac{2A}{\mu_i}\left[\{(-\nu\mu_i^2/R^2)\exp(-\nu\mu_i^2/R^2)\,t + (\nu\mu_i^2/R^2)\cos\omega t + \right.$$

$$\left. + \omega\sin\omega t\}/\{(\nu\mu_i^2/R^2)^2 + \omega^2\}\right].$$

These expressions with (10) show that the effective diffusion coefficient very rapidly acquires a steady-state value for $\lambda_1 = \lambda_2$ and is dependent not only on time.

If the pressure in the system changes stepwise in accordance with the Heaviside function, then for $t \to \infty$ we get from (13) an expression for the effective diffusion coefficient [1].

We made quantitative estimates of the effects of the relaxation characteristics on the diffusion processes by calculating for the flow with parameters $\nu = 4.8\ \mathrm{cm}^2/\mathrm{sec}$, $\lambda_1 = 2.4$ sec; $\lambda_2 = 0.6$; $D_0 = 10^{-5}\ \mathrm{cm}^2/\mathrm{sec}$; $R = 0.2$ cm; $l = 10^3$ cm; the amplitude value of the pressure difference for the three functions was $A = \Delta P_0 = 10^5$ Pa.

Figure 1 shows the numerical results as the relative value of the effective diffusion coefficient D/D_0 in terms of the dimensionless time $\tau = D_0t/R^2$; there is a maximal increase in the mixing rate at the start for the pulse function, with a subsequent fall to zero (curve 1). If a Heaviside function applies, a steady-state value of the mixing rate is established in a relaxing system for a certain time interval (curve 2). For a liquid not having relaxation characteristics, the steady-state values are attained almost instantly (curve 3). A similar situation is observed for a sinusoidal pressure function at frequencies $\omega < 0.01$ sec^{-1} (curves 6 and 7). At frequencies $\omega \geq 1$ sec^{-1}, the rheological characteristics have less effect on the establishment of the steady-state value (curves 4 and 5).

Therefore, the diffusion processes in a flow of relaxing liquid are dependent on the hydrodynamic characteristics and the relaxation ones. One can vary these characteristics to adjust the diffusion in a relaxing medium.

NOTATION

W_m, mean flow velocity; $w(r, t)$, instantaneous flow velocity; R, tube radius; x, coordinate along axis; D_0, molecular diffusion coefficient; D, effective diffusion coefficient; t, time; λ_1, λ_2, relaxation and retardation times; ρ, density; ν, kinematic viscosity; τ, shear stress; $\dot{\gamma}$, shear rate; c, concentration; c_0, axial concentration; c_m, mean concentration; Q, flow rate; $\partial P/\partial x$, pressure gradient; J_0, zero-order Bessel function; J_1, Bessel function of first order, real argument; ξ_i, roots of Bessel function; A, oscillation amplitude; ω, frequency; δ, Dirac delta function; H, Heaviside function.

LITERATURE CITED

1. G. J. Taylor, "Dispersion of soluble matter in a solvent flowing slowly through a tube," Proc. Soc., A219, 186-203 (1953).
2. G. J. Taylor, "The dispersion of matter in turbulent flow through a pipe," Proc. R. Soc., A223, 446-468 (1954).
3. W. N. Gill and R. Sankarasubramanian, "Exact analysis of unsteady convection diffusion," Proc. R. Soc., A316, 341-350 (1970).
4. V. I. Maron, "Propagation of a trace component in a laminar flow in a circular tube," Mekh. Zhidk. Gaza, No. 3, 97-103 (1972).
5. Z. P. Shul'man, Ya. N. Kovalev, and É. A. Zal'tsgendler, Rheophysics of Conglomerate Materials [in Russian], Nauka i Tekhnika, Minsk (1978).
6. P. M. Ogibalov and A. Kh. Mirzadzhanzade, Nonstationary Motions of Viscoplastic Media [in Russian], Moscow State Univ. (1970).
7. H. S. Sharma, "Unsteady flow of viscoeleastic fluids through circular and coaxial circular ducts with pressure gradient as any function of time," Indian J. Pure Appl. Mat., 3, No. 3, 535-542 (1972).
8. D. F. Faizullaev and K. V. Mukuk, Translational Motions of Liquids [in Russian], FAN Uzb. SSR, Tashkent (1976).
9. E. Jahnke, F. Emde, and F. Losch, Special Functions [Russian translation], Nauka, Moscow (1968).